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Exclusion Processes with Degenerate Rates: Convergence to Equilibrium and Tagged Particle[1]

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Stochastic lattice gases with degenerate rates, namely conservative particle systems where the exchange rates vanish for some configurations, have been introduced as simplified models for glassy dynamics. We introduce two particular models and consider them in a finite volume of size ℓ in contact with particle reservoirs at the boundary. We prove that, as for non-degenerate rates, the inverse of the spectral gap and the logarithmic Sobolev constant grow as ℓ^2. It is also shown how one can obtain, via a scaling limit from the logarithmic Sobolev inequality, the exponential decay of a Lyapunov functional for a degenerate parabolic differential equation (porous media equation). We analyze finally the tagged particle displacement for the stationary process in infinite volume. In dimension larger than two we prove that, in the diffusive scaling limit, it converges to a Brownian motion with non-degenerate diffusion coefficient.

KEY WORDS: Exclusion processes; spectral gap; logarithmic Sobolev inequalities; tagged particle diffusion.

1. INTRODUCTION

We analyze some models of stochastic lattice gases with hard core exclusion, i.e., systems of particles on a lattice $\Lambda \subset \mathbb{Z}^d$ with the constraint that on each site there is at most one particle. A configuration is therefore defined by giving for each site $x \in \Lambda$ the occupation number $\eta_x \in \{0, 1\}$, which

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represents an empty or occupied site. The dynamics is given by a continuous time Markov process, which allows the exchange of the occupation numbers across a bond $\{x, y\}$ of neighboring sites x and y with a rate $c_{x,y}(\eta)$ depending on the configuration η. The simplest case is the symmetric simple exclusion process (SEP), in which $c_{x,y}(\eta)=1$. We instead consider processes in which there are some constraints in order for the exchange to be allowed, i.e., the rate $c_{x,y}(\eta)$ degenerates for some configurations η. We mention that Glauber-like, non-conservative, models with degenerate rates have been studied in ref. 1. We shall discuss conservative stochastic lattice gases with degenerate rates both in infinite volume, $\Lambda=\mathbb{Z}^d$, and on a bounded domain. In the latter case we shall however allow exchanges of particles with external reservoirs at the boundary.

This kind of interacting particle systems, called *kinetically constrained lattice gases,* have been introduced in the physical literature as simplified models for some peculiar phenomena of the "glassy" dynamics.[25] Let us recall the physical problem.[10] Experimentally, a glass can be obtained by cooling a liquid fast enough in order to avoid crystallization. Below the melting temperature the liquid enters a metastable phase in which the relaxation time is rather long and increase dramatically if the temperature is further lowered. When the time to reach equilibrium becomes longer than the experimentally accessible time scales, the liquid freezes in an amorphous solid phase, which is called glass. A complete theoretical explanation of this "glass transition" is still lacking.[23] The first question to be settled is whether the glass is a new state of matter or a long lived metastable state, i.e., whether the dramatic increase of the relaxation time is due to an underlying equilibrium transition or is a dynamical phenomenon. Since no static divergent correlation length is detected and the structural properties show a very small temperature dependence,[10] it is possible that a purely dynamical transition takes place. Thus a major goal is to understand the mechanism inducing the *dynamical arrest* which prevents the relaxation of the system. This should be related to other phenomena such as the stretched exponential decay of the structure function for temperatures close to the glass transition and the aging phenomena for supercooled liquids quenched to lower temperatures.[8,29]

In this context, several approaches have been proposed. We mention the *random first order scenario* [17,18] and the *mode coupling theory* [13]; from them the following picture in the mean field approximation arises.[8] There exists a finite temperature T_d such that for $T > T_d$ the configuration space consists of a single ergodic component, while for $T < T_d$ it is broken into many disconnected ergodic components. Moreover the relaxation time diverges as $T \downarrow T_d$. An open question is how to go beyond this mean field picture, in particular to establish whether this ergodic/non-ergodic transition is an effect of the mean field approximation or is related to the behavior of real glasses. The kinetically constrained particle systems, originally introduced to get some insight on the physical mechanism for the dynamical arrest,(19) have recently(4,30) also been used to investigate how the mean field scenario has to be modified for short range models.

The basic idea behind these models is that the motion of a molecule is inhibited by geometrical constraints due to the presence of surrounding particles. In particular, a molecule can be caged by neighboring ones and the cage must be opened to allow its motion. It is thus possible that these local constraints might produce, for a finite value of the density, a cooperative behavior inducing the slowing down of the dynamics. The kinetically constrained lattice gases, are therefore defined by choosing exchange rates $c_{x,y}$ which encode this cage effect. Despite their simplicity and the discrete character, they might capture, at least at a mesoscopic level, some of the key dynamical ingredients of real glasses. Moreover, they have recently been used(26) to model granular systems, which is another class of systems displaying a glass-like dynamical arrest, known as jamming transition.(14,21)

One of the most studied among such models is the so called Kob–Andersen model (KA).(19) Let $m=1,\dots,2d-1$, the KA model is defined by choosing $c_{x,y}(\eta)=1$ if at least m nearest–neighbors of x different from y are empty and m nearest–neighbors of y different from x are also empty, $c_{x,y}(\eta)=0$ otherwise. Note that for $m=0$ we would recover the symmetric simple exclusion. Let us consider this model for $d\geqslant 2$ in a finite cube $\Lambda\subset\mathbb{Z}^d$ of size ℓ; it satisfies the detailed balance w.r.t. the uniform measure on the hyperplanes with fixed total number of particles. However, it is not ergodic, for instance if $d=2$ and $m=1$ two fully occupied consecutive rows do not move. By the general theory of finite state Markov chain it is however possible to decompose the state space into irreducible components. A natural question is the asymptotic behavior, as $\ell\to\infty$, of the probability $p_\ell(\rho)$ of the "maximal irreducible component", here ρ is the density of particles on the hyperplane. The ergodic/non-ergodic transition at a density $\rho_c\in[0,1]$ would correspond to $p_\infty(\rho)=1$ for $\rho<\rho_c$ and $p_\infty(\rho)<1$ for $\rho>\rho_c$. An alternative definition is obtained by looking at the model directly in infinite volume. We note the process satisfies detailed balance w.r.t. each Bernoulli product measure μ_ρ, $\rho\in[0,1]$ but there are other invariant measures, for instance some concentrated on single configurations. Denoting by P_t the semigroup associated to the process, by the spectral theorem, we have that

$$\lim_{t\to\infty}\int d\mu_\rho(\eta)\left[P_t f(\eta)-\mu_\rho(f)\right]^2=0 \quad \text{for any } f\in L_2(\mu_\rho)$$

if and only if zero is a simple eigenvalue of the generator; in such a case we say that the process is ergodic in $L_2(\mu_\rho)$. If the process is ergodic for $\rho < \widehat{\rho}_c$ and not ergodic for $\rho > \widehat{\rho}_c$ we would then say that the ergodic/non-ergodic transition occurs at $\widehat{\rho}_c$.

In the case $d = 3$ and $m = 2$, numerical simulations of the KA model[19] suggested that such a transition, in the sense of the former definition, takes place at $\rho_c \simeq 0.881$. However, in ref. 30 it is shown that for $d = 2$, $m = 1$ and $d = 3$, $m = 1, 2$ we have $\rho_c = 1$ whereas for $d = 2$, $m = 2, 3$ and $d = 3$, $m = 3, 4, 5$ we have $\rho_c = 0$. The finite size corrections are also discussed in ref. 30; in particular it shown that for ρ close to one the thermodynamic limit is achieved for sizes of order $L(\rho) = \exp\{c(1-\rho)^{-1}\}$ for $d = 2$, $m = 1$ and $d = 3$, $m = 1$ whereas $L(\rho) = \exp\{\exp\{c(1-\rho)^{-1}\}\}$ for $d = 3$, $m = 2$.

It is also possible to consider the KA model in a finite volume of size ℓ in contact with particle reservoirs at the boundary (see refs. 4 and 20) for numerical simulations. The total number of particles is not anymore conserved and, for $m \leqslant d - 1$, the system is ergodic in the whole configuration space and reversible w.r.t. the Bernoulli measure whose density is fixed by the reservoirs. However, the dynamical arrest is not ruled out since, as $\ell \to \infty$, the speed of convergence toward the unique invariant measure might exhibit a crossover as a function of the density. A preliminary question is then the asymptotic behavior of the relaxation time, which might be defined as the inverse of the spectral gap of the generator, for $\ell \to \infty$. Note that for SEP the relaxation time grows as ℓ^2 uniformly in the density.

Another physically relevant issue is the asymptotic displacement $x(t)$ of a tagged particle. Indeed, for supercooled liquids near the glass transition, a decrease of the mean square displacement $\mathbb{E}\big(x(t)^2\big)$ is experimentally detected[10] with the possibility of a crossover from a diffusive, to a sub-diffusive behavior related to the dynamical arrest. Let us denote by $D_{\text{self}} = D_{\text{self}}(\rho)$ the diffusion coefficient of a tagged particle. For the KA model with $d = 3$ and $m = 2$ the numerical simulation in ref. 19 for $\rho < \rho_c \simeq 0.881$ suggested the power law behavior $D_{\text{self}} = (\rho_c - \rho)^\alpha$, with $\alpha \simeq 0.3$. This evidence is however due to finite size effects; in fact in ref. 31 it is proved that for each $\rho \in [0, 1)$ the diffusion coefficient is strictly positive.

The main results of this paper are sharp asymptotics on the relaxation time and the diffusive behavior of the tagged particle for some kinetically constrained lattice gases. The first model is defined by an exchange rate $c_{x,y}$ which vanishes if the two neighbors of the bond $\{x, y\}$ along the direction $y - x$ are both occupied. In the second model the exchange rate $c_{x,y}$ across the bond $\{x, y\}$ vanishes if more than half of its neighboring sites, i.e., more than $2d-1$ neighbors, are occupied. These models are in the same spirit of KA, but the degeneracy of the rates is not comparable. Indeed, there are exchanges allowed for KA and forbidden for our models as well as the converse. There is however an important simplifying feature of the models defined above, which plays an essential role in the rigorous analysis: it is possible to construct a finite cluster of empty sites which, uniformly in the configuration on its complement, can be shifted using only allowed exchanges. The results of this paper show that the models above defined behave essentially as the simple exclusion process and therefore they are not really appealing as models for the glass transition. The asymptotic as $\rho \uparrow 1$ of the relaxation time and of $D_{\text{self}}(\rho)$ are however different from SEP. Of course, an interesting issue is whether the results obtained in this paper holds also for the KA model, in other words if the simplifying feature mentioned above is only a technical need or the behavior of KA is essentially different. As argued in ref. 30, 31, to which we refer for a further discussion on this point, it is expected that also KA is ergodic in $L_2(\mu_\rho)$ and the tagged particle diffusion is not degenerate for each $\rho \in [0, 1)$. The asymptotic of relaxation times as $\ell \to \infty$ and the behavior of $D_{\text{self}}(\rho)$ as $\rho \uparrow 1$ might however be different; their analysis appears to be a more difficult task.

Another relevant issue is the macroscopic behavior of the kinetically constrained lattice gases. For non-degenerate rates, the hydrodynamical limit[28] states that if the initial condition has a density profile then, under a diffusive rescaling, at later times we still have a density profile which can be obtained from the initial one by solving a parabolic equation. For SEP this is simply the heat equation. If the rates degenerate a natural candidate for the hydrodynamic limit is a parabolic equation of porous media type degenerating when the density approaches one; an analogous result has indeed been proven for a model in which the occupation number η_x is a continuous variable.[12] For kinetically constrained lattice gases there is however a serious obstruction: the hydrodynamic limit cannot hold for any initial condition admitting a density profile. Consider for instance the models described above in $d=1$ and take an initial configuration given by a sequence of two occupied sites and one empty at the left of the origin and three occupied sites and one empty at the right. This configuration is invariant for the microscopic dynamic, however the associated density profile evolves diffusively for the putative macroscopic evolution. Indeed, the initial density profile is 2/3 at the left and 3/4 at the right, it is thus bounded away from 1 and therefore is not affected by the degeneracy. On the other hand such phenomenon is somehow exceptional and we expect a hydrodynamic behavior for a suitable large class of initial conditions.

Outline and Summary of Results

In Section 2 we define more precisely the models we analyze and introduce the basic notation.

In Sections 3 we consider these models on finite volume of size ℓ with reservoirs allowing particle exchanges at the boundary. By our choice of the rates the processes are ergodic and reversible w.r.t. the product Bernoulli measure whose density is fixed by the reservoirs. We then discuss the rate of convergence to this unique invariant measure. In particular we show that the spectral gap shrinks to zero as ℓ^{-2} when $\ell \to \infty$. An analogous result for the simple exclusion process on the hyperplane with fixed number of particles has been obtained in refs. 24, Section 8 and has been extended to Kawasaki dynamics, under suitable mixing conditions on the invariant measure, in refs. 6 and 22. Our proof is based on a comparison argument with the Glauber dynamics reversible w.r.t. the same Bernoulli measure and the construction of a suitable path which enables to move a particle from the boundary to any site by using only allowed exchanges. This technique allows us also to show that the logarithmic Sobolev constant, which controls the exponential decay of the entropy, grows as ℓ^2. An analogous result for the Kawasaki dynamics, under suitable mixing conditions on the invariant measure, has been obtained in refs. 7 and 31.

In Section 4 we consider these processes on infinite volume, in such a case they are reversible w.r.t. the Bernoulli measure μ_ρ and we show they are ergodic in $L_2(\mu_\rho)$ for any $\rho \in [0, 1]$. We do not discuss the rate of convergence to equilibrium in this context, which for the exclusion process is algebraic with diffusive exponent $d/2$ (see refs. 5, 11 and 15). We analyze the displacement $x(t)$ of a tagged particle for the stationary process. We recall that for SEP in $d = 1$ we have[3] $\mathbb{E}\big(x(t)^2\big) \approx \sqrt{t}$ while for $d \geqslant 2$ the displacement $x(t)$ satisfies[16] a central limit theorem with strictly positive variance, i.e., $\mathbb{E}\big(x(t)^2\big) \approx t$. By the ergodicity of the process in $L_2(\mu_\rho)$, one can repeat the arguments in refs. 16 and 28 and show that, under a diffusive rescaling, $x(t)$ converges to a Brownian motion with diffusion coefficient $D_{\rm self}(\rho)$, which is given by a variational formula. Of course, if $d = 1$ we have that $D_{\rm self}(\rho) = 0$ as for the simple exclusion. By following the strategy in refs. 27 and 28 we finally prove that for $d \geqslant 2$ and each $\rho \in [0, 1)$ we have $D_{\rm self}(\rho) > 0$.

Finally, in Section 5, we show how one can take a scaling limit of the logarithmic Sobolev inequality. By this procedure we obtain the exponential decay of a Lyapunov functional for the porous media equation.

2. DEFINITION OF THE MODELS

The spatial structure is modeled by the d-dimensional cubic lattice $\mathbb{Z}^d$ in which we let e_i, $i = 1, \dots, d$ be the coordinate unit vectors. We denote by x, y, z the sites of $\mathbb{Z}^d$ and by $\mathrm{d}(x, y) := |x - y|$ the Euclidean distance between x and y. Given $A, B \subset \mathbb{Z}^d$ we then let, as usual, $\mathrm{d}(A, B) = \inf\{\mathrm{d}(x, y)\,,\, x \in A, y \in B\}$.

For $A \subset \mathbb{Z}^d$, the configuration space in A is $\Omega_A := \{0, 1\}^A$. If $A = \mathbb{Z}^d$, we drop it from the notation, namely we let $\Omega := \Omega_{\mathbb{Z}^d}$. We can regard a configuration $\eta \in \Omega_A$ as a map from A to $\{0, 1\}$; the value $\eta_x \in \{0, 1\}$ is interpreted as the number of particles of the configuration η at the site x. If $\eta \in \Omega_A$ and $B \subset A$ we denote by η_B the restriction of the configuration η to Ω_B. Let $A, B \subset \mathbb{Z}^d$ be disjoint subsets, $A \cap B = \emptyset$; given $\eta \in \Omega_A$ and $\xi \in \Omega_B$ we let $\eta\xi \in \Omega_{A\cup B}$ be the configuration such that $(\eta\xi)_A = \eta$ and $(\eta\xi)_B = \xi$. For $\eta \in \Omega$ and $x \in \mathbb{Z}^d$ we denote by $\vartheta_x \eta$ the configuration η shifted by x, namely $(\vartheta_x \eta)_y := \eta_{y-x}$, $y \in \mathbb{Z}^d$. Given a probability measure μ and a random variable f we denote by $\mu(f)$ the expectation of f w.r.t. μ and by $\mu(f; f) := \mu\big(f - \mu(f)\big)^2$ its variance.

For $\eta \in \Omega$ we let $T_{x,y}\eta \equiv \eta^{x,y}$ be the configuration obtained from η by exchanging the number of particles in x and y, i.e.,

$$(T_{x,y}\eta)_z := \begin{cases} \eta_y & \text{if } \ z = x, \\ \eta_x & \text{if } \ z = y, \\ \eta_z & \text{if } \ z \neq x, y. \end{cases} \tag{2.1}$$

Analogously, we let $T_x \eta \equiv \eta^x$ be the configuration obtained from η by flipping the occupation number in x, i.e.,

$$(T_x \eta)_z := \begin{cases} 1 - \eta_x & \text{if } \ z = x, \\ \eta_z & \text{if } \ z \neq x. \end{cases} \tag{2.2}$$

We let also T_x, respectively, $T_{x,y}$, act on functions $f : \Omega \to \mathbb{R}$ as $T_x f(\eta) := f(T_x \eta)$, respectively $T_{x,y} f(\eta) := f(T_{x,y}\eta)$. We finally introduce $\nabla_x f := T_x f - f$ and $\nabla_{x,y} f := T_{x,y} f - f$.

The models we consider are defined as follows. Given a positive integer ℓ, let $\Lambda := [1, \ell]^d \cap \mathbb{Z}^d$, $\Lambda^{\mathrm{c}} = \mathbb{Z}^d \setminus \Lambda$. We consider the continuous time Markov process on the configuration space Ω_Λ with generator

$$L_\Lambda := L_{\text{bulk}} + \frac{1}{\ell} L_{\text{bound}}, \tag{2.3}$$

where L_{bulk} describes the exchanges of particles in bulk; it is given by

$$L_{\text{bulk}} f(\eta) = \sum_{\substack{\{x,y\}\subset\Lambda \\ \mathrm{d}(x,y)=1}} c_{x,y}(\eta) \nabla_{x,y} f(\eta) \tag{2.4}$$

namely the occupation numbers at the sites $x, y \in \Lambda$ are exchanged with rate $c_{x,y}$. On the other hand $L_{\text{bound.}}$ describes the effect of the particle reservoirs at the boundary of Λ; it is given by

$$L_{\text{bound.}} f(\eta) = \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ \mathrm{d}(x,y)=1}} c_x(\eta) \nabla_x f(\eta). \tag{2.5}$$

For $\rho \in (0, 1)$ and $x \in \Lambda$ we choose the flip rates $c_x(\eta)$ as

$$c_x(\eta) := (1-\rho)\eta_x + \rho(1-\eta_x). \tag{2.6}$$

Therefore, $L_{\text{bound.}}$ is the generator of the following process. A particle on the interior boundary of Λ leaves the system with rate $1-\rho$ (in fact at higher rate at the corners) while particles enter the system, if the landing site is empty, with rate ρ. We emphasize that in (2.3) the boundary part of the dynamics is slowed by a factor $1/\ell$, as we prove below this is the minimal choice to obtain that the relaxation time diverges as ℓ^2.

We let $0 \in \Omega$ be the configuration in which all the sites are empty. We shall discuss two specific choices of the exchange rates $c_{x,y}(\eta)$, $\eta \in \Omega_\Lambda$. The first is

$$c^{(1)}_{x,x+e_i}(\eta) := \begin{cases} 0 & \text{if } \ (\eta 0_{\Lambda^c})_{x-e_i} + (\eta 0_{\Lambda^c})_{x+2e_i} = 2, \qquad i = 1, \ldots, d, \\ 1 & \text{otherwise} \end{cases} \tag{2.7}$$

namely the exchange across the bond $\{x, x+e_i\}$ is suppressed if the neighboring sites in the i direction are both occupied. Note that, since $(\eta 0_{\Lambda^c})_x = 0$ for $x \notin \Lambda$, exchanges across the bonds $\{x, x+e_i\}$ such that either $x - e_i \in \Lambda^c$ or $x + 2e_i \in \Lambda^c$ are not suppressed.

The second is

$$c^{(2)}_{x,y}(\eta) := \begin{cases} 0 & \text{if } \displaystyle\sum_{z:\, \mathrm{d}(\{z\},\{x,y\})=1} (\eta 0_{\Lambda^c})_z > 2d-1, \\ 1 & \text{otherwise} \end{cases} \tag{2.8}$$

namely the exchange across the bond $\{x, y\}$ is suppressed if more than one half of the neighboring sites are occupied. Note that for $d = 1$ we have $c^{(1)}_{x,y} = c^{(2)}_{x,y}$. We shall denote by $L^{(1)}_\Lambda$, respectively, $L^{(2)}_\Lambda$ the generator (2.3) with c_x chosen as in (2.6) and $c_{x,y} = c^{(1)}_{x,y}$, respectively, $c_{x,y} = c^{(2)}_{x,y}$.

Let $\mu_{\Lambda,\rho}$ be the Bernoulli measure on Ω_Λ with density ρ, i.e., $\mu_{\Lambda,\rho}$ is the product measure on Ω_Λ with marginal $\mu_{\Lambda,\rho}(\eta_x = 1) = \rho$. It is easy to check that the generator $L^{(k)}_\Lambda$, $k = 1, 2$, is self-adjoint in $L_2(\mu_{\Lambda,\rho})$, equivalently the rates satisfy detailed balance w.r.t. $\mu_{\Lambda,\rho}$.

We note that the bulk dynamics L_{bulk} preserves the total number of particles in Λ, but—since the rates degenerate—it is not ergodic on all the hyperplanes of Ω_Λ with fixed total number of particles. For instance, if $d=1$, all configurations η in which the distance between all the empty sites is three or more do not evolve. On the other hand, thanks to $L_{\text{bound.}}$, it is not difficult to show that the generator $L_\Lambda^{(k)}$, $k=1,2$, is irreducible, namely there is positive probability of going from any configuration to any other. By standard theory on finite state space Markov chain, irreducibility of $L_\Lambda^{(k)}$ implies the uniqueness of the invariant measure and that 0 is a simple eigenvalue of $L_\Lambda^{(k)}$. In Section 3 we prove a lower bound on the spectral gap of $L_\Lambda^{(k)}$ in $L_2(\mu_{\Lambda,\rho})$ showing that for each $\rho\in(0,1)$ it shrinks to 0 as ℓ^{-2}.

In order to discuss the diffusive behavior of the tagged particle we need to introduce also the infinite volume dynamics. The configuration space is then $\Omega=\{0,1\}^{\mathbb{Z}^d}$, a function $f:\Omega\longrightarrow\mathbb{R}$ is called a *local* function if it depends only on finitely many η_x. The generator of the process acts on local functions as

$$\mathcal{L}^{(k)} f(\eta)=\sum_{\substack{\{x,y\}\subset\mathbb{Z}^d\\ \mathrm{d}(x,y)=1}} c_{x,y}^{(k)}(\eta)\nabla_{x,y} f(\eta), \tag{2.9}$$

where $c^{(k)}$, $k=1,2$, has been defined in (2.7),(2.8), where now we have $\Lambda^{\mathrm{c}}=\emptyset$. Note that $c^{(k)}$ are translationally covariant in the sense that $c_{x+y,x+y+e}^{(k)}(\vartheta_y\eta)=c_{x,x+e}^{(k)}(\eta)$. Moreover, for each $\rho\in[0,1]$ and $k=1,2$, the generator $\mathcal{L}^{(k)}$ is self adjoint in $L_2(\mu_\rho)$, where μ_ρ is the Bernoulli measure in Ω with density ρ. In Section 4 we prove that, for each $\rho\in[0,1]$ and $k=1,2$, the generator $\mathcal{L}^{(k)}$ is ergodic in $L_2(\mu_\rho)$, namely that 0 is a simple eigenvalue.

We consider the process $\eta(t)$ generated by $\mathcal{L}^{(k)}$ and condition that at time zero the origin is occupied; we then tag the particle at the origin and denote by $x(t)$ its position at time t. The pair $\big(\eta(t),x(t)\big)$ is then a Markov process on the state space $\big\{(\eta,x)\in\Omega\times\mathbb{Z}^d : \eta_x=1\big\}$ with generator

$$\begin{aligned}\mathcal{A}^{(k)}F(\eta,x) :=& \sum_{\substack{y\in\mathbb{Z}^d\\ \mathrm{d}(x,y)=1}} c_{x,y}^{(k)}(\eta)(1-\eta_y)\big[F(\eta^{x,y},y)-F(\eta,x)\big]\\ &+\sum_{\substack{\{y,z\}\subset\mathbb{Z}^d\setminus\{x\}\\ \mathrm{d}(y,z)=1}} c_{y,z}^{(k)}(\eta)\big[F(\eta^{y,z},x)-F(\eta,x)\big].\end{aligned} \tag{2.10}$$

Let $\Omega_0 := \{\eta \in \Omega : \eta_0 = 1\}$ and $\mu_{\rho,0}$ be the Bernoulli measure on Ω_0 with marginal $\mu_{\rho,0}(\eta_x = 1) = \rho,\ x \in \mathbb{Z}^d \setminus \{0\}$. We shall consider the process $\big(\eta(t), x(t)\big)$ generated by $\mathcal{A}^{(k)}$ with initial condition $x(0)=0$ and $\eta(0)$ distributed according to $\mu_{\rho,0}$. In Section 4, for $d \geqslant 2$, we prove the invariance principle for the position of the tagged particle, namely that $\varepsilon x(\varepsilon^{-2}t)$ converges in distribution, as $\varepsilon \to 0$, to a Brownian motion with strictly positive diffusion coefficient.

3. SPECTRAL GAP AND LOG-SOBOLEV INEQUALITY

The spectral gap of the Markov generator L_Λ is defined as

$$\mathrm{gap}(L_\Lambda) := \inf \mathrm{spec}\,(-L_\Lambda \restriction \mathbb{1}^\perp),$$

where $\mathbb{1}^\perp$ is the subspace of $L_2(\mu_{\Lambda,\rho})$ orthogonal to the constant functions. Since Ω_Λ is finite, by irreducibility of L_Λ, we trivially have $\mathrm{gap}(L_\Lambda) > 0$, we next discuss its asymptotic behavior as $\ell \to \infty$.

In the case of non-degenerate rates $c_{x,y} = 1$, namely for the symmetric simple exclusion process, the aforementioned problem of non ergodicity on hyperplanes is not present. Let $\nu_{\Lambda,n}(\eta) := \mu_{\Lambda,\rho}\big(\eta \big| \sum_{x\in\Lambda} \eta_x = n\big)$ be the canonical measure with n particles. In refs. 24, Section 8 it is proven that, considering L_{bulk} with $c_{x,y} = 1$ on $L_2(\nu_{\Lambda,n})$, we have $\mathrm{gap}(L_{\mathrm{bulk}}) \asymp \ell^{-2}$ uniformly in n, here $a_\ell \asymp b_\ell$ means there exists a constant $C > 0$ such that $C^{-1} b_\ell \leqslant a_\ell \leqslant C b_\ell$ for any $\ell > 0$. In the case $c_{x,y} = 1$ it is not difficult to prove, and in fact it is a corollary of our analysis, that also for L_Λ on $L_2(\mu_{\Lambda,\rho})$ we have $\mathrm{gap}(L_\Lambda) \asymp \ell^{-2}$ uniformly in ρ.

Our first results is a lower bound on the spectral gap of $L_\Lambda^{(k)}$, $k = 1, 2$. Let us define the Dirichlet form associated to $L_\Lambda^{(k)}$ as

$$\begin{aligned}
\mathcal{E}^{(k)}_{\Lambda,\rho}(f) &:= -\mu_{\Lambda,\rho}(f L^{(k)}_\Lambda f) \\
&= \frac{1}{2}\Bigg\{ \sum_{\substack{\{x,y\}\subset\Lambda \\ \mathrm{d}(x,y)=1}} \mu_{\Lambda,\rho}\big[c^{(k)}_{x,y} (\nabla_{x,y} f)^2\big] \\
&\qquad + \frac{1}{\ell} \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ \mathrm{d}(x,y)=1}} \mu_{\Lambda,\rho}\big[c_x (\nabla_x f)^2\big] \Bigg\}.
\end{aligned} \tag{3.1}$$

Theorem 3.1. For each $\rho \in (0, 1)$ and $k = 1, 2$ there exists a constant $C = C(d, \rho, k)$ such that for any ℓ and for any function $f : \Omega_\Lambda \to \mathbb{R}$ we have

$$\mu_{\Lambda,\rho}(f; f) \leqslant C\, \ell^2\, \mathcal{E}^{(k)}_{\Lambda,\rho}(f). \tag{3.2}$$

Remark 1. Thanks to the variational characterization of the spectral gap, the bound (3.2) is equivalent to $\mathrm{gap}(L_\Lambda^{(k)}) \geqslant C^{-1}\ell^{-2}$. Moreover, by letting $P_t^{(k)} := \exp\{tL_\Lambda^{(k)}\}$ be the semigroup generated by $L_\Lambda^{(k)}$, we also have that (3.2) is equivalent to

$$\| P_t^{(k)} f - \mu_{\Lambda,\rho} f \|_{L_2(\mu_{\Lambda,\rho})} \leqslant \exp\left\{ -\frac{t}{C\ell^2} \right\} \| f - \mu_{\Lambda,\rho} f \|_{L_2(\mu_{\Lambda,\rho})}$$

for any function f on Ω_Λ. Finally, let $\mathbb{E}_{\mu_{\Lambda,\rho}}^{(k)}$ be the distribution of the stationary process generated by $L_\Lambda^{(k)}$, i.e., we take $\mu_{\Lambda,\rho}$ as the starting measure of the Markov chain. Then (3.2) is equivalent to

$$\mathbb{E}_{\mu_{\Lambda,\rho}}^{(k)}\big(f(\eta(0));\, f(\eta(t))\big) \leqslant \exp\left\{ -\frac{t}{C\ell^2} \right\} \| f - \mu_{\Lambda,\rho} f \|^2_{L_2(\mu_{\Lambda,\rho})}$$

for any function f on Ω_Λ.

Remark 2. By taking as test function $f(\eta)=\sum_{x\in\Lambda}(\eta_x-\rho)\sin\pi(x-1)/\ell$ and using $c_{x,y}^{(k)} \leqslant 1$ for any x, y and $k=1,2$, a simple computation shows that for each $\rho \in (0,1)$ there exist a constant $C = C(d,k,\rho)$ such that $\mathrm{gap}(L_\Lambda^{(k)}) \leqslant C\ell^{-2}$. Hence $\mathrm{gap}(L_\Lambda^{(k)}) \asymp \ell^{-2}$ as in the case of the simple exclusion.

Remark 3. As discussed in Section 1, the correct dependence of the spectral gap on the density ρ has some interest. For simplicity we discuss it only in the case $k=1$, namely for the rates chosen as in (2.7). It is a corollary of our analysis that the gap goes to zero as $\rho \uparrow 1$ as a power law of exponent between 1 and 2. More precisely the following two inequalities hold. There exists a constant $C_1 = C_1(d)$ such that for any $\rho \in (0,1)$ and any ℓ we have $\mathrm{gap}(L_\Lambda^{(1)}) \geqslant (1-\rho)^2 C_1/\ell^2$. For each integer $\ell \geqslant 5$, there exists a constant $C_2 = C_2(d,\ell)$ such that $\mathrm{gap}(L_\Lambda^{(1)}) \leqslant C_2(1-\rho)$. The lower bound follows from the proof of Theorem 3.1; the upper bound is obtained easily by using as test function $f(\eta) = \eta_x$ with $x \in \Lambda$ such that $\mathrm{d}(x,\Lambda^{\mathrm{c}}) \geqslant 3$.

Remark 4. As previously discussed, the process generated by L_{bulk} is not irreducible on the hyperplanes with fixed number of particles, $\Omega_{\Lambda,N} := \big\{\eta \in \Omega_\Lambda : \sum_{x\in\Lambda} \eta_x = N\big\}$. In the one-dimensional case, $d=1$ (recall that in such a case $c^{(1)} = c^{(2)}$), is however not difficult to check that L_{bulk} is irreducible on the set

$$\widetilde{\Omega}_{\Lambda,N} := \big\{\eta \in \Omega_{\Lambda,N} : \exists\, x, y \in \Lambda,\ x \neq y,\ \mathrm{d}(x,y) \leqslant 2 \ \text{ such that } \eta_x = \eta_y = 0\big\}$$

and L_{bulk} satisfies detailed balance w.r.t. the conditional measure $\widetilde{\nu}_{\Lambda,N}(\cdot) := \mu_{\Lambda,\rho}(\cdot|\widetilde{\Omega}_{\Lambda,N})$. A natural question is then the asymptotic behavior of the spectral gap of L_{bulk} in $L_2(\widetilde{\nu}_{\Lambda,N})$, a reasonable guess is that for each $N \leqslant \ell - 2$ we still have $\text{gap}(L_{\text{bulk}}) \asymp \ell^{-2}$. This conjecture is supported by the fact that if $N = |\Lambda| - 2 = \ell - 2$, i.e., in the highest density case, we have a single pair of neighboring empty sites which performs a random walk. We are not able to prove the above conjecture in general, but only in the trivial situation in which $N < |\Lambda|/3 = \ell/3$. In such a case, for ℓ large enough, we have $\widetilde{\Omega}_{\Lambda,N} = \Omega_{\Lambda,N}$; therefore, the statement follows easily by a comparison with the exclusion process with long exchanges (see ref. 24, Lemma 8.1) and a minor modification of the argument in Lemma 3.3.

We recall that a Markov process with reversible measure μ and Dirichlet form $\mathcal{E}$ is said to satisfy a logarithmic Sobolev inequality with constant c_{LS} iff for any function f we have

$$\mu\left(f^2 \log \frac{f^2}{\mu(f^2)} \right) \leqslant c_{\text{LS}}\, \mathcal{E}(f). \tag{3.3}$$

It is well known, (see e.g., ref. 2), that (3.3) implies the hypercontractivity of the semigroup P_t associated to the Markov process and the exponential decay of the entropy, namely, for any probability density f w.r.t. μ we have

$$\mu\Big(P_t f \,\log\, P_t f\Big) \leqslant \exp\left\{ -\frac{4t}{c_{\text{LS}}} \right\} \mu\big(f \log f\big).$$

The logarithmic Sobolev inequalities for the process generated by $L_\Lambda^{(k)}$, $k = 1, 2$, is stated as follows.

Theorem 3.2. For each $k = 1, 2$ and $\rho \in (0, 1)$ there exists a constant $C = C(d, k, \rho)$ such that for any ℓ and for any function $f : \Omega_\Lambda \to \mathbb{R}$ we have

$$\mu_{\Lambda,\rho}\left(f^2 \log \frac{f^2}{\mu_{\Lambda,\rho}(f^2)} \right) \leqslant C\, \ell^2\, \mathcal{E}_{\Lambda,\rho}^{(k)}(f). \tag{3.4}$$

The key step in the proof of Theorems 3.1 and 3.2 is the following bound, which is in the same spirit of the path lemmata in refs. 6, 7, 22, 24, 31.

Lemma 3.3. For $k = 1, 2$ and each $x \in \Lambda$ let

$$U_x^{(k)} := \big\{ y \in \Lambda : 1 \leqslant y_1 \leqslant x_1 - 1,\ |y_i - x_i| \leqslant r^{(k)},\ i = 2, \ldots, d \big\},$$

where $r^{(1)} := 0$ and $r^{(2)} := 1$. Then, for each $k = 1, 2$ and $\rho \in (0, 1)$, there exists a constant $A = A(d, k, \rho)$ such that for any ℓ

$$\mu_{\Lambda,\rho}\big(c_x(\nabla_x f)^2\big) \leqslant A\Big\{\ell \sum_{y\in U_x^{(k)}} \mu_{\Lambda,\rho}\big(c^{(k)}_{y,y+e_1}(\nabla_{y,y+e_1} f)^2\big) + \sum_{\substack{y:\, y_1=1\\ y\in U_x^{(k)}}} \mu_{\Lambda,\rho}\big(c_y(\nabla_y f)^2\big)\Big\} \tag{3.5}$$

for any $x \in \Lambda$ and any function f on Ω_Λ.

Postponing the proof of the lemma above, let us first show how it implies, together with a comparison argument with Glauber dynamics, Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Let us introduce the product (Glauber) dynamics in Ω_Λ defined by the generator

$$L^G_\Lambda f(\eta) := \sum_{x\in\Lambda} c_x(\eta)\nabla_x f(\eta), \tag{3.6}$$

where c_x has been defined in (2.6). The generator L^G_Λ is self-adjoint in $L_2(\mu_{\Lambda,\rho})$; since it is a product dynamics, it is immediate to check its spectral gap is 1. For each function f on Ω_Λ we thus get

$$\begin{aligned}\mu_{\Lambda,\rho}(f; f) &\leqslant -\mu_{\Lambda,\rho}(f L^G_\Lambda f) = \tfrac{1}{2}\mathrm{A}\ \textstyle\sum_{x\in\Lambda} \mu_{\Lambda,\rho}\big(c_x(\nabla_x f)^2\big)\\ &\leqslant \tfrac{1}{2}\mathrm{A}\textstyle\sum_{x\in\Lambda}\Big\{\ell \sum_{y\in U_x^{(k)}} \mu_{\Lambda,\rho}\big(c^{(k)}_{y,y+e_1}(\nabla_{y,y+e_1} f)^2\big)\\ &\quad + \displaystyle\sum_{\substack{y:\, y_1=1\\ y\in U_x^{(k)}}} \mu_{\Lambda,\rho}\big(c_y(\nabla_y f)^2\big)\Big\}\\ &\leqslant \tfrac{1}{2}\mathrm{A}\ell(2r^{(k)}+1)^{d-1}\Big\{\ell \sum_{\substack{\{x,y\}\subset\Lambda\\ \mathrm{d}(x,y)=1}} \mu_{\Lambda,\rho}\big(c^{(k)}_{x,y}(\nabla_{x,y} f)^2\big)\\ &\quad + \sum_{\substack{x\in\Lambda,\, y\notin\Lambda\\ \mathrm{d}(x,y)=1}} \mu_{\Lambda,\rho}\big(c_x(\nabla_x f)^2\big)\Big\},\end{aligned} \tag{3.7}$$

where we used the variational characterization of the spectral gap, Lemma 3.3 and elementary inequalities. Recalling (3.1), we then get the bound (3.2) with $C = A(2r^{(k)}+1)^{d-1}$. ∎

Proof of Theorem 3.2. Let L^G_Λ be the generator on Ω_Λ introduced in (3.6); then, (see e.g., ref. 2), for any $\rho \in (0,1)$ it satisfies the logarithmic Sobolev inequality (3.3) with c_{LS} given by $C_1(\rho) := (1-2\rho)^{-1} \log[(1-\rho)/\rho]$ (we understand $C_1(1/2)=2$) uniformly in ℓ. By (3.5) and (3.7) the bound (3.4), with $C = C_1(\rho)\, A\, (2r^{(k)}+1)^{d-1}$ follows. ∎

We are left with the proof of Lemma 3.3. The basic idea is to use first $L_{\text{bound.}}$ to empty a few sites at the boundary. Then—via careful moves—we show how this cluster of holes can be shifted, using exchanges with non-zero rate, and used to flip the occupation number in x. Finally we shift the cluster back to the boundary and use again $L_{\text{bound.}}$ to reconstruct the initial configuration near the boundary.

Proof of Lemma 3.3. We discuss first the case of $k=1$ which corresponds to the rates (2.7). We assume also that $x_1 \geqslant 4$, otherwise the proof is much easier. The path we shall construct is schematically depicted in Fig. 1 which is meant to help the reader in following the steps detailed in the sequel.

Given $\eta \in \Omega_\Lambda$ and $x \in \Lambda$ let us define $S_x\eta \in \Omega_\Lambda$ as the configuration given by

$$(S_x\eta)_y := \begin{cases} 0 & \text{if } \; y = (1, x_2, \dots, x_d), \\ 0 & \text{if } \; y = (2, x_2, \dots, x_d), \\ 1 - \eta_x & \text{if } \; y = (3, x_2, \dots, x_d), \\ \eta_y & \text{otherwise.} \end{cases} \tag{3.8}$$

Moreover, for $y \in \mathbb{Z}^d$ we define

$$\begin{aligned} T^L_y &:= T_{y+e_1, y+2e_1} T_{y, y+e_1} T_{y-e_1, y}, \\ T^R_y &:= T_{y, y+e_1} T_{y+e_1, y+2e_1} T_{y+2e_1, y+3e_1}. \end{aligned} \tag{3.9}$$

Note that T^L_y moves the occupation number in $y-e_1$ to $y+2e_1$ while the configuration in $y, y+e_1, y+2e_1$ is shifted by one in the direction $-e_1$. Analogously T^R_y moves the occupation number in $y+3e_1$ to y while the configuration in $y, y+e_1, y+2e_1$ is shifted by one in the direction e_1.

For $x \in \Lambda$, we let $\gamma_i := (ie_1, x_2, \dots, x_d)$, $i = 1, \dots x_1 - 1$ and define

$$\overline{S}_x\eta := \big(T^L_{\gamma_2} \cdots T^L_{\gamma_{x_1-3}}\big) T_{x-e_1, x} \big(T^R_{\gamma_{x_1-4}} \cdots T^R_{\gamma_1}\big) S_x\eta. \tag{3.10}$$

It is not difficult to check that $(\overline{S}_x\eta)_y = 0$ if $y = \gamma_1$ or $y = \gamma_2$, $(\overline{S}_x\eta)_y = \eta_x$ if $y = \gamma_3$, $(\overline{S}_x\eta)_y = 1 - \eta_x$ if $y = x$, and $(\overline{S}_x\eta)_y = \eta_y$ otherwise.

For $x \in \Lambda$ with $x_1 \geqslant 4$ we write

$$\nabla_x f(\eta) = \big[f(\eta^x) - f(\overline{S}_x\eta)\big] + \big[f(\overline{S}_x\eta) - f(S_x\eta)\big] + \big[f(S_x\eta) - f(\eta)\big]. \tag{3.11}$$

We start by considering the second term in decomposition above; we claim that

$$\begin{aligned}\sum_{\eta\in\Omega_\Lambda} \mu_{\Lambda,\rho}(\eta) c_x(\eta)\big[f(\overline{S}_x\eta) - f(S_x\eta)\big]^2 &\leqslant 18(1-\rho)^{-2}(2x_1-7)\\ &\quad\times \sum_{i=1}^{x_1-1} \mu_{\Lambda,\rho}\Big[c^{(1)}_{\gamma_i,\gamma_{i+1}}\big(\nabla_{\gamma_i,\gamma_{i+1}} f\big)^2\Big].\end{aligned} \tag{3.12}$$

To prove the above bound, let us introduce the path (see Fig. 1)

$$\zeta_i := \begin{cases} S_x\eta & i=0,\\ T^R_{\gamma_i}\cdots T^R_{\gamma_1}S_x\eta & i=1,\dots,x_1-4,\\ T_{x-e_1,x}\big(T^R_{\gamma_{x_1-4}}\cdots T^R_{\gamma_1}\big)S_x\eta & i=x_1-3,\\ \big(T^L_{\gamma_{2x_1-5-i}}\cdots T^L_{\gamma_{x_1-3}}\big)T_{x-e_1,x}\big(T^R_{\gamma_{x_1-4}}\cdots T^R_{\gamma_1}\big)S_x\eta & i=x_1-3+1,\dots,2x_1-7.\end{cases} \tag{3.13}$$

By (3.10) we have $\zeta_{2x_1-7} = \overline{S}_x\eta$. Note also that, for each $\eta\in\Omega_\Lambda$ and $0\leqslant i\leqslant x_1-4$, the configuration ζ_i is guaranteed to be empty at the sites γ_{i+1} and γ_{i+2}. Moreover, for each $\eta\in\Omega_\Lambda$ and $0\leqslant j\leqslant x_1-4$ the configuration ζ_{x_1-3+j} is guaranteed to be empty at the sites γ_{x_1-2-j} and γ_{x_1-3-j}. This will allow us to move the configuration ζ_i to the configuration ζ_{i+1} using exchanges with non zero $c^{(1)}$ rate (see Fig. 1).

By telescopic sums and Cauchy–Schwartz, we get

$$\begin{aligned}\big[f(\overline{S}_x\eta) - f(S_x\eta)\big]^2 &= \Bigg(\sum_{i=1}^{2x_1-7}\big[f(\zeta_i) - f(\zeta_{i-1})\big]\Bigg)^2\\ &\leqslant (2x_1-7)\sum_{i=1}^{2x_1-7}\big[f(\zeta_i) - f(\zeta_{i-1})\big]^2.\end{aligned} \tag{3.14}$$

We consider only the case $1\leqslant i\leqslant x_1-4$, the others are analogous. We then have $\zeta_i = T^R_{\gamma_i}\zeta_{i-1}$; recalling (3.9), again by telescopic sums and

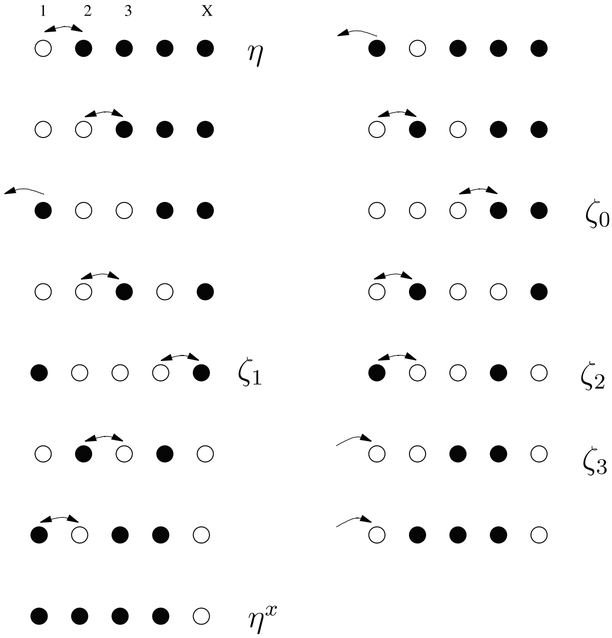

Fig. 1. *Example of the path constructed to prove Lemma 3.3 for* $k=1$ (see Definition 3.13, here we are drawing the one-dimensional case and $x_1=5$). ○ denotes empty sites, ● denote occupied sites and arrows denote the exchange done in the next move.

Cauchy–Schwartz, we get

$$\begin{aligned}
&\left[f(\zeta_i)-f(\zeta_{i-1})\right]^2\\
&=\left[f(T^R_{\gamma_i}\zeta_{i-1})-f(\zeta_{i-1})\right]^2\\
&\leqslant 3\Big\{\left[f(T_{\gamma_i,\gamma_i+e_1}T_{\gamma_i+e_1,\gamma_i+2e_1}T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\right.\\
&\qquad\left.-f(T_{\gamma_i+e_1,\gamma_i+2e_1}T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\right]^2\\
&\qquad+\left[f(T_{\gamma_i+e_1,\gamma_i+2e_1}T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})-f(T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\right]^2\\
&\qquad+\left[f(T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})-f(\zeta_{i-1})\right]^2\Big\}\\
&=3\Big\{c^{(1)}_{\gamma_i,\gamma_i+e_1}(T_{\gamma_i+e_1,\gamma_i+2e_1}T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\\
&\qquad\left[\nabla_{\gamma_i,\gamma_i+e_1}f(T_{\gamma_i+e_1,\gamma_i+2e_1}T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\right]^2\\
&\qquad+c^{(1)}_{\gamma_i+e_1,\gamma_i+2e_1}(T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\left[\nabla_{\gamma_i+e_1,\gamma_i+2e_1}f(T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})\right]^2\\
&\qquad+c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}(\zeta_{i-1})\left[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1}f(\zeta_{i-1})\right]^2\Big\},
\end{aligned}\tag{3.15}$$

where we used that

$$c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}(\zeta_{i-1}) = c^{(1)}_{\gamma_i+e_1,\gamma_i+2e_1}(T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1})$$
$$= c^{(1)}_{\gamma_i,\gamma_i+e_1}(T_{\gamma_i+e_1,\gamma_i+2e_1}T_{\gamma_i+2e_1,\gamma_i+3e_1}\zeta_{i-1}) = 1$$

by construction of the path ζ_i, see the remark below (3.13).

In order to prove the bound (3.12) we consider only the last term on the r.h.s. of (3.15), the other can be analyzed in the same way. Given $A \subset \Lambda$, $\xi \in \Omega_A$, $x \in A$, and $g : \Omega_\Lambda \to \mathbb{R}$ we observe that, since $\mu_{\Lambda,\rho}$ is a product measure and c_x is given in (2.6),

$$\sum_{\eta\in\Omega_\Lambda} \mu_{\Lambda,\rho}(\eta)\,\eta_x\,c_x(\eta)\,g(\eta_{\Lambda\setminus A}\,\xi) = \sum_{\eta\in\Omega_\Lambda} \mu_{\Lambda,\rho}(\eta)\,(1-\eta_x)\,c_x(\eta)\,g(\eta_{\Lambda\setminus A}\,\xi)$$
$$= \rho(1-\rho)\mu_{\Lambda,\rho}\big(g\,\big|\,\eta_A = \xi\big). \tag{3.16}$$

Recalling definitions (3.8) and (3.13), we thus get

$$\begin{aligned}
&\sum_{\eta\in\Omega} \mu_{\Lambda,\rho}(\eta)c_x(\eta)c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}(\zeta_{i-1})\big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1}f(\zeta_{i-1})\big]^2\\
&\quad= \sum_{\eta\in\Omega} \mu_{\Lambda,\rho}(\eta)\eta_x c_x(\eta)c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}(\zeta_{i-1})\big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1}f(\zeta_{i-1})\big]^2\\
&\qquad+ \sum_{\eta\in\Omega} \mu_{\Lambda,\rho}(\eta)[1-\eta_x]c_x(\eta)c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}(\zeta_{i-1})\big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1}f(\zeta_{i-1})\big]^2\\
&\quad= \rho(1-\rho)\Big\{\mu_{\Lambda,\rho}\Big(c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}\big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1}f\big]^2\\
&\qquad\qquad\Big|\,\eta_x = 1,\, \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = \eta_{\gamma_{i+2}} = 0\Big)\\
&\qquad\qquad+\mu_{\Lambda,\rho}\Big(c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}\big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1}f\big]^2\\
&\qquad\qquad\Big|\,\eta_x = \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = 0,\, \eta_{\gamma_{i+2}} = 1\Big)\Big\}.
\end{aligned}\tag{3.17}$$

We now observe that for any positive function $g : \Omega_\Lambda \to \mathbb{R}$ we have

$$\begin{aligned}
\mu_{\Lambda,\rho}(g) \geqslant\ &\mu_{\Lambda,\rho}(\eta_x = 1,\, \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = \eta_{\gamma_{i+2}} = 0)\\
&\times\mu_{\Lambda,\rho}\big(g\,\big|\,\eta_x = 1,\, \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = \eta_{\gamma_{i+2}} = 0\big)\\
&+\mu_{\Lambda,\rho}(\eta_x = \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = 0,\, \eta_{\gamma_{i+2}} = 1)\\
&\times\mu_{\Lambda,\rho}\big(g\,\big|\,\eta_x = \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = 0,\, \eta_{\gamma_{i+2}} = 1\big)
\end{aligned}$$

$$= \rho(1-\rho)^3 \Big\{ \mu_{\Lambda,\rho}\big(g \,\big|\, \eta_x = 1,\, \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = \eta_{\gamma_{i+2}} = 0\big) \\ + \mu_{\Lambda,\rho}\big(g \,\big|\, \eta_x = \eta_{\gamma_i} = \eta_{\gamma_{i+1}} = 0,\, \eta_{\gamma_{i+2}} = 1\big)\Big\}, \tag{3.18}$$

so that from (3.17) we get

$$\sum_{\eta\in\Omega_\Lambda} \mu_{\Lambda,\rho}(\eta) c_x(\eta) c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}(\zeta_{i-1}) \big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1} f(\zeta_{i-1})\big]^2 \\ \leqslant (1-\rho)^{-2}\, \mu_{\Lambda,\rho}\Big(c^{(1)}_{\gamma_i+2e_1,\gamma_i+3e_1}\big[\nabla_{\gamma_i+2e_1,\gamma_i+3e_1} f\big]^2\Big). \tag{3.19}$$

The bound (3.12) follows from (3.14), (3.15), and (3.19). Note that the extra factor 2 comes from the return path, $x_1 - 4 \leqslant i \leqslant 2x_1 - 7$.

Let us now consider the last term on the r.h.s. of (3.11); we claim that

$$\sum_{\eta\in\Omega} \mu_{\Lambda,\rho} c_x(\eta)\big[f(S_x\eta) - f(\eta)\big]^2 \\ \leqslant \frac{6}{(1-\rho)^2}\Big\{3\mu_{\Lambda,\rho}\big(c_{\gamma_1}[\nabla_{\gamma_1} f]^2\big) + 2\mu_{\Lambda,\rho}\big(c_{\gamma_1,\gamma_2}[\nabla_{\gamma_1,\gamma_2} f]^2\big) \\ + \mu_{\Lambda,\rho}\big(c_{\gamma_2,\gamma_3}[\nabla_{\gamma_2,\gamma_3} f]^2\big)\Big\}. \tag{3.20}$$

To prove the above bound let us define $T_x^+\eta$ as the configuration given by $(T_x^+\eta)_x = 1$ and $(T_x^+\eta)_y = \eta_y$ for $y \neq x$; analogously we let $T_x^-\eta$ be the configuration given by $(T_x^-\eta)_x = 0$ and $(T_x^-\eta)_y = \eta_y$ for $y \neq x$. Recalling (3.8) we then have

$$\begin{aligned} f(S_x\eta) - f(\eta) = \eta_x\Big[&\eta_{\gamma_1}\nabla_{\gamma_1} f(\eta) + \nabla_{\gamma_1,\gamma_2} f(T^-_{\gamma_1}\eta) + \eta_{\gamma_2}\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta) \\ &+ \nabla_{\gamma_2,\gamma_3} f(T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta) + \nabla_{\gamma_1,\gamma_2} f(T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta) \\ &+ \eta_{\gamma_3}\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)\Big] \\ + (1-\eta_x)\Big[&(1-\eta_{\gamma_1})\nabla_{\gamma_1} f(\eta) \\ &+ \nabla_{\gamma_1,\gamma_2} f(T^+_{\gamma_1}\eta) + \eta_{\gamma_2}\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta) \\ &+ \nabla_{\gamma_2,\gamma_3} f(T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta) \\ &+ \nabla_{\gamma_1,\gamma_2} f(T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta) \\ &+ \eta_{\gamma_3}\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)\Big]. \end{aligned} \tag{3.21}$$

By using Schwartz inequality and the fact that the telescopic decomposition in (3.21) has been arranged so that all the exchanges have non zero $c^{(1)}$ rate, we get

$$
\begin{aligned}
[f(S_x\eta)-f(\eta)]^2 \leqslant 6\Big\{ & \eta_x \ \eta_{\gamma_1}[\nabla_{\gamma_1} f(\eta)]^2+(1-\eta_x)(1-\eta_{\gamma_1})[\nabla_{\gamma_1} f(\eta)]^2 \\
& +\eta_x c^{(1)}_{\gamma_1,\gamma_2}(T^-_{\gamma_1}\eta)[\nabla_{\gamma_1,\gamma_2} f(T^-_{\gamma_1}\eta)]^2 \\
& +(1-\eta_x)c^{(1)}_{\gamma_1,\gamma_2}(T^+_{\gamma_1}\eta)[\nabla_{\gamma_1,\gamma_2} f(T^+_{\gamma_1}\eta)]^2 \\
& +\eta_x\eta_{\gamma_2}[\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)]^2 \\
& +(1-\eta_x)\eta_{\gamma_2}[\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)]^2 \\
& +\eta_x c^{(1)}_{\gamma_2,\gamma_3}(T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)[\nabla_{\gamma_2,\gamma_3} f(T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)]^2 \\
& +(1-\eta_x)c^{(1)}_{\gamma_2,\gamma_3}(T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)[\nabla_{\gamma_2,\gamma_3} f(T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)]^2 \\
& +\eta_x c^{(1)}_{\gamma_1,\gamma_2}(T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta) \\
& \times[\nabla_{\gamma_1,\gamma_2} f(T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)]^2 \\
& +(1-\eta_x)c^{(1)}_{\gamma_1,\gamma_2}(T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta) \\
& \times[\nabla_{\gamma_1,\gamma_2} f(T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)]^2 \\
& +\eta_x\eta_{\gamma_3}[\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)]^2 \\
& +(1-\eta_x)\eta_{\gamma_3}[\nabla_{\gamma_1}(f(T_{\gamma_1,\gamma_2}T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)]^2\Big\}.
\end{aligned} \tag{3.22}
$$

By recalling the definition (2.6) we have $1-\eta_y \leqslant \rho^{-1}c_y(\eta)$ and $\eta_y \leqslant (1-\rho)^{-1}c_y(\eta)$ for any $\eta \in \Omega_\Lambda$ and $y \in \Lambda$. We next estimate separately each term on the r.h.s. of (3.22). Let us consider only the last two terms, the others are easier. We have

$$
\begin{aligned}
\sum_{\eta\in\Omega_\Lambda} \mu_{\Lambda,\rho}(\eta)\ c_x(\eta)\ & \Big\{\eta_x\ \eta_{\gamma_3}[\nabla_{\gamma_1} f(T_{\gamma_1,\gamma_2}T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^-_{\gamma_1}\eta)]^2 \\
& \quad +(1-\eta_x)\ \eta_{\gamma_3}[\nabla_{\gamma_1}(f(T_{\gamma_1,\gamma_2}T_{\gamma_2,\gamma_3}T^-_{\gamma_1}T_{\gamma_1,\gamma_2}T^+_{\gamma_1}\eta)]^2\Big\} \\
\leqslant (1-\rho)^{-1}\ & \Big\{(1-\rho)\ \mu_{\Lambda,\rho}(\eta_x)\mu_{\Lambda,\rho}\big(c_{\gamma_1}[\nabla_{\gamma_1} f]^2 \,\big|\, \eta_x=1,\,\eta_{\gamma_2}=\eta_{\gamma_3}=0\big) \\
& \quad +\rho\ \mu_{\Lambda,\rho}(1-\eta_x)\mu_{\Lambda,\rho}\big(c_{\gamma_1}[\nabla_{\gamma_1} f]^2 \,\big|\, \eta_x=\eta_{\gamma_2}=0,\,\eta_{\gamma_3}=1\big)\Big\} \\
\leqslant (1-\rho)^{-2}\ & \mu_{\Lambda,\rho}\big(c_{\gamma_1}[\nabla_{\gamma_1} f]^2\big),
\end{aligned} \tag{3.23}
$$

where we used that, as in (3.18), for any positive function $g:\Omega_\Lambda\to\mathbb{R}$

$$
\begin{aligned}
& \mu_{\Lambda,\rho}\big(g \,\big|\, \eta_x=1,\,\eta_{\gamma_2}=\eta_{\gamma_3}=0\big)+\mu_{\Lambda,\rho}\big(g \,\big|\, \eta_x=\eta_{\gamma_2}=0,\,\eta_{\gamma_3}=1\big) \\
& \quad \leqslant \frac{1}{\rho(1-\rho)^2}\,\mu_{\Lambda,\rho}\big(g\big).
\end{aligned}
$$

By analogous computations for the other terms, (3.20) follows.

Finally, to bound the first term on the r.h.s. of (3.11), it is enough to change variable $\eta \mapsto \eta^x$. Indeed, noting $\bar{S}_x \eta^x = S_x \eta$, we get

$$\sum_{\eta \in \Omega_\Lambda} \mu_{\Lambda,\rho}(\eta) c_x(\eta) \big[f(\eta^x) - f(\bar{S}_x \eta) \big]^2 \\ = \sum_{\eta \in \Omega_\Lambda} \mu_{\Lambda,\rho}(\eta) c_x(\eta) \big[f(\eta) - f(S_x \eta) \big]^2. \tag{3.24}$$

For x such that $x_1 \geqslant 4$, the bound (3.5), with the constant A given by $A = 180(1-\rho)^{-2}$ now follows from (3.11), (3.12), (3.20), and (3.24). The case in which $1 \leqslant x_1 \leqslant 3$ can be proven directly by the same steps leading to (3.20).

In the case $k=2$, namely for the choice (2.8) of the exchange rates, we give only a rough sketch of the proof, which is very similar to the case $k=1$. Indeed, it is enough to define the configuration $S_x \eta$, analogous to (3.8), as

$$(S_x \eta)_y := \begin{cases} 0 & \text{if} \quad y \in Q_1((3, x_2, \ldots, x_d)), \\ 1 - \eta_x & \text{if} \quad y = (1, x_2, \ldots, x_d), \\ \eta_y & \text{otherwise}, \end{cases} \tag{3.25}$$

where $Q_1(x) := \{ y \in \mathbb{Z}^d : \max_{i=1,\ldots,d} |x_i - y_i| \leqslant 1 \}$ is the cube of side three centered in x. The configuration $S_x \eta$ can then be shifted using exchanges with non zero $c^{(2)}$ rates by means of a suitable path, depicted in Fig. 2 for

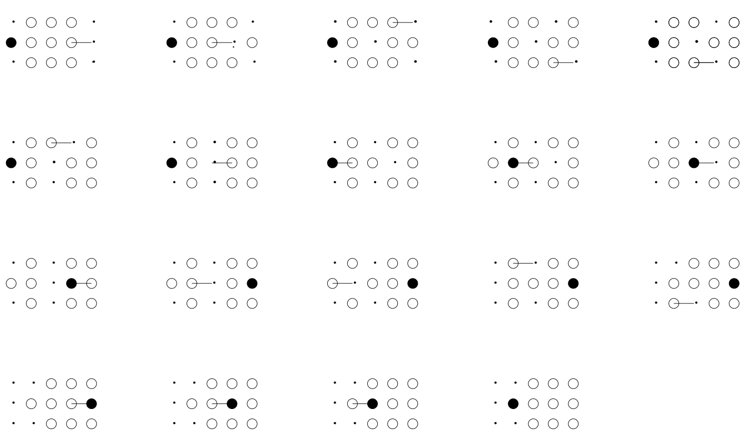

Fig. 2. *The shifting path.* ○ denotes sites guaranteed empty, ● denotes $1 - \eta_x$, · denotes an arbitrary occupation number, and—denotes the bond exchanged in the next move.

$d = 2$. The proof is finally completed by the same arguments given for the choice (2.7). ∎

4. Diffusion of the tagged particle

In this section we consider stochastic lattice gases with degenerate rates in infinite volume. We first prove that, for each $\rho \in [0, 1]$, the generator $\mathcal{L}^{(k)}$, $k = 1, 2$ defined in (2.9) is ergodic in $L_2(\mu_\rho)$, recall that μ_ρ is the Bernoulli measure on Ω.

Proposition 4.1. For each $\rho \in [0, 1]$ and $k = 1, 2$ we have that zero is a simple eigenvalue of the generator $\mathcal{L}^{(k)}$ considered on $L_2(\mu_\rho)$.

Proof. Let

$$\mathcal{E}_\rho^{(k)}(f) = \frac{1}{2} \sum_{\substack{\{x,y\} \subset \mathbb{Z}^d \\ \mathrm{d}(x,y)=1}} \mu_\rho\big[c_{x,y}^{(k)}(\nabla_{x,y} f)^2\big] \tag{4.1}$$

be the Dirichlet form of the generator $\mathcal{L}^{(k)}$. To show zero is a simple eigenvalue of $\mathcal{L}^{(k)}$ we check that $\mathcal{E}_\rho^{(k)}(f) = 0$ implies f constant μ_ρ–a.s. This is trivially true for $\rho = 0, 1$. For $\rho \in (0, 1)$, by De Finetti's theorem, it is enough to show that $\mathcal{E}_\rho^{(k)}(f) = 0$ implies $\mu_\rho(\nabla_{x,y} f)^2 = 0$ for each $\{x, y\} \subset \mathbb{Z}^d$ with $\mathrm{d}(x, y) = 1$.

We discuss in some detail the case $k = 1$. Let $x \in \mathbb{Z}^d$ and consider the bond $\{x, x + e_i\}$, $i = 1, \ldots, d$. For $n = 1, 2, \ldots$ we introduce the events

$$B_{x,i}^n := \big\{\eta \in \Omega\, :\, \eta_{x+ne_i} = \eta_{x+(n+1)e_i} = 0\big\}, \qquad B_{x,i} := \bigcup_{n \geqslant 1} B_{x,i}^n.$$

By noting that $\mu_\rho(B_{x,i}) = 1$, we have

$$\mu_\rho(\nabla_{x,x+e_i} f)^2 = \mu_\rho\big([\nabla_{x,x+e_i} f]^2\, \mathbb{1}_{B_{x,i}}\big) \leqslant \sum_{n=1}^{\infty} \mu_\rho\big([\nabla_{x,x+e_i} f]^2\, \mathbb{1}_{B_{x,i}^n}\big). \tag{4.2}$$

Let $\gamma_h := x + he_i$, $h = 0, 1, \ldots$; given $\eta \in B_{x,i}^n$ we can find a path $\eta = \zeta_0, \ldots, \zeta_N = \eta^{x,x+e_i}$, where $\zeta_{j+1} = \zeta_j^{\gamma_h,\gamma_{h+1}}$ for some $h = 0, 1, \ldots$ and $c_{\gamma_h,\gamma_{h+1}}^{(1)}(\zeta_j) = 1$. It is in fact possible to construct a path analogous to the one introduced in the proof of Lemma (3.3); note that the two sites γ_n and

γ_{n+1} are empty by the definition of the event $B^n_{x,i}$. Since $\mathcal{E}^{(1)}_\rho(f)=0$ implies

$$\mu_\rho\big(c^{(1)}_{\gamma_h,\gamma_{h+1}}[\nabla_{\gamma_h,\gamma_{h+1}} f]^2\big)=0 \qquad \text{for any } h=0,1,\dots$$

by telescopic sums and Cauchy–Schwartz in (4.2) we get $\mu_\rho(\nabla_{x,x+e_i} f)^2=0$.

Recalling Fig. (2) it is straightforward to modify the argument given above to cover also the case $k=2$. ∎

We next discuss the diffusive behavior of a tagged particle. More precisely, we consider the process $\big(\eta(t),x(t)\big)$ with generator given in (2.10), initial condition $x(0)=0$ and $\eta(0)$ distributed according to $\mu_{\rho,0}$, the Bernoulli measure on $\Omega_0=\{\eta\in\Omega : \eta_0=1\}$. Let $\xi(t):=\vartheta_{-x(t)}\eta(t)$ be the process as seen from the tagged particle, we have that $\xi(t)$ is itself a Markov process on the configuration space Ω_0 with generator given by

$$\begin{aligned}\mathcal{A}^{(k)}_0 f(\xi) = &\sum_{\substack{y\in\mathbb{Z}^d\\ \mathrm{d}(0,y)=1}} c^{(k)}_{0,y}(\xi)(1-\xi_y)\big[f(\vartheta_{-y}\xi^{0,y})-f(\xi)\big]\\ &+\sum_{\substack{\{x,y\}\subset\mathbb{Z}^d\setminus\{0\}\\ \mathrm{d}(x,y)=1}} c^{(k)}_{x,y}(\xi)\nabla_{x,y} f(\xi).\end{aligned} \tag{4.3}$$

A straightforward computation shows that $\mathcal{A}^{(k)}_0$ is self-adjoint in $L_2(\mu_{\rho,0})$. In the case $d\geqslant 2$, $\mathcal{A}^{(k)}_0$ is also ergodic in $L_2(\mu_{\rho,0})$, as can be established by an argument analogous to the one in Proposition 4.1. We omit the details, see however, Lemma 4.3 where we construct explicitly the appropriate paths to move the configuration around the tagged particle. Thanks to the ergodicity, we can apply the same proof as the one given in refs. 16 and 28 for non-degenerate rates. We get that the rescaled position of the tagged particle, $\varepsilon x(\varepsilon^{-2}t)$, converges in distribution, as $\varepsilon\to 0$, to a d-dimensional Brownian motion with diffusion matrix $2\,D^{(k)}_{\mathrm{self}}(\rho)$. Furthermore the diffusion matrix $D^{(k)}_{\mathrm{self}}(\rho)$ is given by the variational formula

$$\begin{aligned} r\cdot D^{(k)}_{\mathrm{self}}(\rho) r = \frac{1}{2}\inf_{f \text{ local}}\int \mu_{\rho,0}(d\xi)\Big\{&\sum_{\substack{y\in\mathbb{Z}^d\\ \mathrm{d}(0,y)=1}} c^{(k)}_{0,y}(\xi)(1-\xi_y)\\ &\times\big[r\cdot y+f(\vartheta_{-y}\xi^{0,y})-f(\xi)\big]^2\\ &+\sum_{\substack{\{x,y\}\subset\mathbb{Z}^d\setminus\{0\}\\ \mathrm{d}(x,y)=1}} c^{(k)}_{x,y}(\xi)[\nabla_{x,y} f(\xi)]^2\Big\}\end{aligned} \tag{4.4}$$

where $r\in\mathbb{R}^d$ and $\cdot$ is the inner product in $\mathbb{R}^d$.

The main result of this Section is that, for $d \geqslant 2$ and each $\rho \in [0,1)$, the diffusion matrix $D^{(k)}_{\rm self}(\rho)$ is strictly positive as in the case of simple exclusion.

Theorem 4.2. For each $d \geqslant 2$, $k = 1, 2$ and $\rho \in [0,1)$ there exists a real $c = c(d,k,\rho) > 0$ such that $r \cdot D^{(k)}_{\rm self}(\rho) r \geqslant c\, r \cdot r$ for any $r \in \mathbb{R}^d$.

Remark. As discussed in the Section 1, the behavior of $D^{(k)}_{\rm self}(\rho)$ as $\rho \uparrow 1$ has some interest. Note that for SEP it vanishes linearly. In the case $k = 1$, Theorem (4.2) will be proven with $c = c_0(1-\rho)^{11}$ where c_0 does not depend on ρ. However, by the same strategy and some extra efforts, it is possible to improve the lower bound to $c = c_0'(1-\rho)^4$. An upper bound of the form $D^{(k)}_{\rm self}(\rho) \leqslant C_0(1-\rho)^2 \mathbb{1}$ is easily obtained by using a constant test function f in the variational expression (4.4).

Let us fix a direction in $\mathbb{R}^d$, say e_1, and define the following subsets of $\mathbb{Z}^d \setminus \{0\}$

$$
\begin{aligned}
R^{(1)}_0 &:= \big\{x \in \mathbb{Z}^d \setminus \{0\} : \max_{i=1,2} |x_i| = 1\,,\ x_i = 0\,, i = 3, \dots, d\big\},\\
R^{(1)}_{\pm 1} &:= \big\{x \in \mathbb{Z}^d \setminus \{0\} : x_1 = \pm 2\,,\ |x_2| \leqslant 1\,,\ x_i = 0\,, i = 3, \dots, d\big\}
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
R^{(2)}_0 &:= \big\{x \in \mathbb{Z}^d \setminus \{0\} : x_1 = 0\,,\ \max_{i=2,\dots,d} |x_i| \leqslant 3\big\},\\
R^{(2)}_{\pm 1} &:= \big\{x \in \mathbb{Z}^d \setminus \{0\} : x_1 = \pm 1\,,\ \max_{i=2,\dots,d} |x_i| \leqslant 3\big\}.
\end{aligned}
\tag{4.6}
$$

Given $\xi \in \Omega_0$, we next define $\xi^{+,-,(k)}$ as the configuration obtained from ξ by exchanging the occupation numbers in $R^{(k)}_{+1}$ with the corresponding ones in $R^{(k)}_{-1}$, namely

$$
\big(\xi^{+,-,(1)}\big)_x := \begin{cases} \xi_x & \text{if } \ x \notin R^{(1)}_{+1} \cup R^{(1)}_{-1}, \\ \xi_{x \mp 4e_1} & \text{if } \ x \in R^{(1)}_{\pm 1} \end{cases}
\tag{4.7}
$$

and

$$
\big(\xi^{+,-,(2)}\big)_x := \begin{cases} \xi_x & \text{if } \ x \notin R^{(2)}_{+1} \cup R^{(2)}_{-1}, \\ \xi_{x \mp 2e_1} & \text{if } \ x \in R^{(2)}_{\pm 1}. \end{cases}
\tag{4.8}
$$

We finally introduce the events

$$\mathcal{B}_{\pm}^{(k)} := \{\xi \in \Omega_0 : \xi_{R_0^{(k)}} = 0\,,\ \xi_{R_{\pm 1}^{(k)}} = 0\} \qquad \mathcal{B}^{(k)} := \mathcal{B}_+^{(k)} \cup \mathcal{B}_-^{(k)} \tag{4.9}$$

and note that $\xi \in \mathcal{B}_+^{(k)}$ iff $\xi^{+,-,(k)} \in \mathcal{B}_-^{(k)}$.

Lemma 4.3. For each $d \geqslant 2$, $k = 1, 2$ and $\rho \in [0, 1)$ there exists a real $a = a(d, k, \rho) > 0$ such that for any $r \in \mathbb{R}^d$ we have

$$\begin{aligned} r \cdot D_{\text{self}}^{(k)}(\rho) r \geqslant (r \cdot e_1)^2 \frac{a}{2} \inf_{f \text{ local}} \int \mu_{\rho,0}\big(d\xi \,\big|\, \mathcal{B}^{(k)}\big) \Big\{ \big[f(\xi^{+,-,(k)}) - f(\xi)\big]^2 \\ + \sum_{y=\pm 1} \mathbb{1}_{\{\xi_{R_y^{(k)}} = 0\}}(\xi) \big[y + f(\vartheta_{-ye_1} \xi^{0,ye_1}) - f(\xi)\big]^2 \Big\}. \end{aligned} \tag{4.10}$$

Proof. We discuss in some detail the case $k = 1$. We note that if $\xi \in \mathcal{B}^{(1)}$ and $y = \pm 1$ we have

$$c_{0,ye_1}^{(1)}(\xi)[1 - \xi_{ye_1}] \geqslant \mathbb{1}_{\{\xi_{R_y^{(1)}} = 0\}}(\xi),$$

since $\mu_{\rho,0}(\mathcal{B}^{(1)}) \geqslant (1 - \rho)^{11}$, by the same argument as in (3.18), from (4.4) we then get

$$\begin{aligned} r \cdot D_{\text{self}}^{(1)}(\rho) r \ &\geq (r \cdot e_1)^2 \tfrac{1}{2} \inf_{f \text{ local}} \\ &\times \Big\{ \int \mu_{\rho,0}(d\xi) \sum_{\substack{\{x,y\} \subset \mathbb{Z}^d \setminus \{0\} \\ \mathrm{d}(x,y) = 1}} c_{x,y}^{(1)}(\xi) [\nabla_{x,y} f(\xi)]^2 + (1 - \rho)^{11} \\ &\times \int \mu_{\rho,0}\big(d\xi \,\big|\, \mathcal{B}^{(1)}\big) \sum_{y=\pm 1} \mathbb{1}_{\{\xi_{R_y^{(1)}} = 0\}}(\xi) \\ &\times \big[y + f(\vartheta_{-ye_1} \xi^{0,ye_1}) - f(\xi)\big]^2 \Big\}. \end{aligned} \tag{4.11}$$

Let $T_1, \ldots, T_{16}$ be the chain of exchanges depicted in Fig. 3, T_i exchanges the occupation numbers in the bond b_i. Note that if $\xi \in \mathcal{B}_+^{(1)}$ the path $\zeta_0^+ := \xi$, $\zeta_1^+ := T_1 \zeta_0^+, \ldots, \zeta_{16}^+ := T_{16} \zeta_{15}^+ = \xi^{+,-,(1)}$ is such that

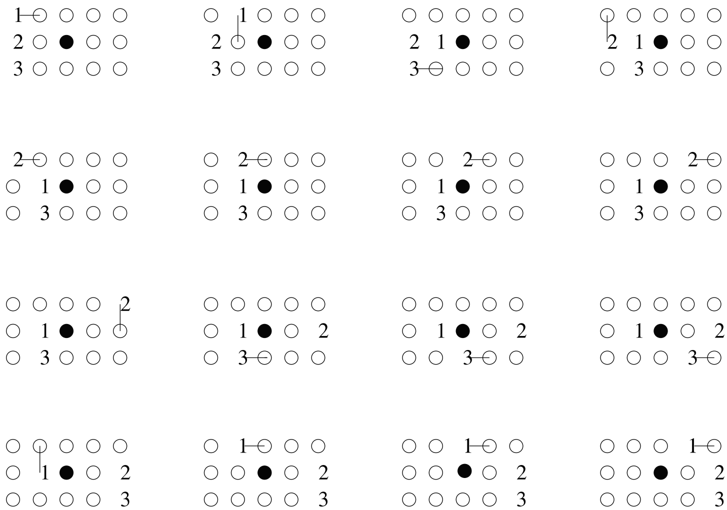

Fig. 3. *Chain of exchanges* $T_1, \ldots, T_{16}$ *for* $k=1$. The picture represents sites in the plane e_1, e_2, ● denotes site 0 and—in the i-th figure denotes the bond exchanged by T_i. If the sites denoted by ○ are empty then the starting configuration is in $\mathcal{B}_+^{(1)}$. In such a case 1,2, and 3 denote the occupation numbers in $R_-^{(1)}$ which, step by step, are moved to $R_+^{(1)}$ by using only allowed exchanges.

$c_{b_i}^{(1)}(\zeta_{i-1}^+)=1,\ i=1,\ldots,16$. For $\xi\in\mathcal{B}_-^{(1)}$ we define analogously $\zeta_0^- := \xi$, $\zeta_1^- := T_{16}\zeta_0^-, \ldots,\ \zeta_{16}^- := T_1\zeta_{15}^- = \xi^{+,-,(1)}$ which is such that $c_{b_{17-i}}^{(1)}(\zeta_{i-1}^-)=1$, $i=1,\ldots,16$.

We then have

$$\begin{aligned}
&\left[f(\xi^{+,-,(1)})-f(\xi)\right]^2 \mathbb{I}_{\mathcal{B}^{(1)}}(\xi)\\
&\quad\le \mathbb{I}_{\mathcal{B}_+^{(1)}}(\xi)\left[f(\xi^{+,-,(1)})-f(\xi)\right]^2+\mathbb{I}_{\mathcal{B}_-^{(1)}}(\xi)\left[f(\xi^{+,-,(1)})-f(\xi)\right]^2\\
&\quad= \mathbb{I}_{\mathcal{B}_+^{(1)}}(\xi)\Big[\sum_{i=1}^{16} f(\zeta_i^+)-f(\zeta_{i-1}^+)\Big]^2\\
&\qquad+\mathbb{I}_{\mathcal{B}_-^{(1)}}(\xi)\Big[\sum_{i=1}^{16} f(\zeta_i^-)-f(\zeta_{i-1}^-)\Big]^2\\
&\quad\le 16\sum_{i=1}^{16}\Big\{c_{b_i}^{(1)}(\zeta_{i-1}^+)\left[\nabla_{b_i} f(\zeta_{i-1}^+)\right]^2+c_{b_{17-i}}^{(1)}(\zeta_{i-1}^-)\left[\nabla_{b_{17-i}} f(\zeta_{i-1}^-)\right]^2\Big\}.
\end{aligned}\tag{4.12}$$

By integrating w.r.t. $\mu_{\rho,0}$ the above inequality and taking into account that in the chain of exchanges T_i, $i=1,\ldots,16$ each bond is used at most twice we get

$$\begin{aligned}
\mu_{\rho,0}\big(\mathcal{B}^{(1)}\big)\int \mu_{\rho,0}\big(d\xi\,\big|\mathcal{B}^{(1)}\big)\left[f(\xi^{+,-,(1)})-f(\xi)\right]^2 \\
= \int \mu_{\rho,0}(d\xi)\left[f(\xi^{+,-,(1)})-f(\xi)\right]^2 \mathrm{1\!I}_{\mathcal{B}^{(1)}}(\xi) \\
\le 64 \int \mu_{\rho,0}(d\xi) \sum_{\substack{\{x,y\}\subset\mathbb{Z}^d\setminus\{0\}\\ \mathrm{d}(x,y)=1}} c^{(1)}_{x,y}(\xi)[\nabla_{x,y} f(\xi)]^2,
\end{aligned} \tag{4.13}$$

which inserted in (4.11) concludes the proof with $a(d,1,\rho)=2^{-6}(1-\rho)^{11}$.

The case $k=2$ is proven by the same arguments; in this case, for $d=2$, the required chain of exchanges $T_1,\dots,T_{35}$ is depicted in Fig. 4. ∎

Proof of Theorem 4.2. Thanks to Lemma 4.3, it is enough to prove that the r.h.s. of (4.10) is strictly positive for $r\cdot e_1 \neq 0$. By the variational formula (4.4), it can be interpreted as the self diffusion coefficient

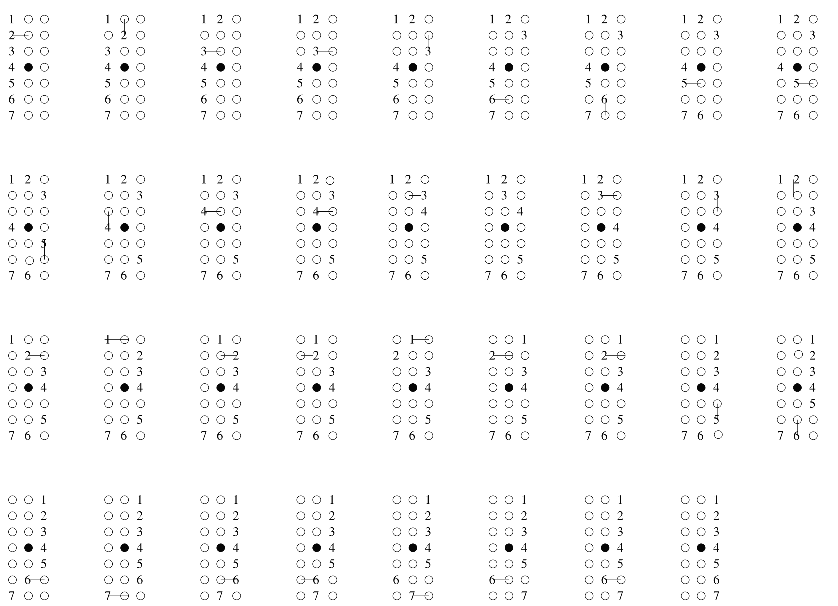

Fig. 4. *Chain of exchanges $T_1,\dots,T_{35}$ for $k=2$.* ● denotes site 0 and—in the i-th figure denotes the bond exchanged by T_i.

of a one dimensional auxiliary process which we next describe in the fixed frame of reference.

The configuration space is $\{(y,\eta)\in\mathbb{Z}\times\Omega : \vartheta_{-ye_1}\eta\in\mathcal{B}^{(k)}\}$. Let $y(t)\in\mathbb{Z}$ be the position of the tagged particle and $\eta(t)$ be the particles configuration. At time $t=0$ the tagged particle is at the origin, $y(0)=0$, and $\eta(0)\in\mathcal{B}^{(k)}$ is distributed according to $\mu_{\rho,0}(\,\cdot\,|\mathcal{B}^{(k)})$. Then the tagged particle jumps to the right, respectively, left, with rate one if $\vartheta_{-y(t)e_1}\eta(t)\in\mathcal{B}^{(k)}_{+}$, respectively, if $\vartheta_{-y(t)e_1}\eta(t)\in\mathcal{B}^{(k)}_{-}$. Moreover, with rate one, $\eta(t)$ is exchanged to $\vartheta_{y(t)e_1}\big[(\vartheta_{-y(t)e_1}\eta(t))^{+,-,(k)}\big]$, namely the occupation numbers in $\vartheta_{y(t)e_1}R^{(k)}_{-}$ are exchanged with the ones in $\vartheta_{y(t)e_1}R^{(k)}_{+}$.

The proof of the theorem can now be completed by showing that there exists a real $c>0$ such that for any $t\geqslant 0$ and $\eta\in\mathcal{B}^{(k)}$ we have $\mathbb{E}_{(0,\eta)}\big(y(t)^2\big)\geqslant ct$. Here $\mathbb{E}_{(0,\eta)}$ denotes the distribution of the auxiliary process with initial condition $(0,\eta)$. This can be proven by the same argument as the one in ref. 28, II.6.3 for the case of non-degenerate rates. Let us denote the initial configuration $\eta(0)$ by η_0. By assumption $\eta_0(0)=1$ and $\eta_0\in\mathcal{B}^{(k)}$. The configurations which can be reached from η_0 are labeled as η_m, $m\in\mathbb{Z}$ and are defined iteratively as

$$\eta_{m+1}=\begin{cases}\eta_m^{y(m),y(m)+e_1} & \text{if } \vartheta_{-y(m)e_1}\eta_m\in\mathcal{B}^{(k)}_{+},\\ \vartheta_{y(m)e_1}\big[(\vartheta_{-y(m)e_1}\eta_m)^{+,-,(k)}\big] & \text{otherwise.}\end{cases}\tag{4.14}$$

$$\eta_{m-1}=\begin{cases}\eta_m^{y(m),y(m)-e_1} & \text{if } \vartheta_{-y(m)e_1}\eta_m\in\mathcal{B}^{(k)}_{-},\\ \vartheta_{y(m)e_1}\big[(\vartheta_{-y(m)e_1}\eta_m)^{+,-,(k)}\big] & \text{otherwise,}\end{cases}\tag{4.15}$$

where $y(m)$ is the position of the tagged particle in the configuration η_m. Note that the above definition is consistent.

Consider a symmetric simple random walk on $\mathbb{Z}$ and denote as $m(t)$ the position of the walker at time t. It is not difficult to check that if $m(0)=0$ then $\eta_{m(t)}=\eta(t)$ for the auxiliary process. More precisely, for any function $f(\eta)$, the expectation value with respect to the probability μ_t evoluted with the generator of the auxiliary process coincides with the expectation with respect to the measure on $m(t)$ generated by the random walk process. It is furthermore immediate to check from definition (4.14), (4.15) that when m changes by 2 the particle position changes by one. Therefore, the following inequality:

$$\mathbb{E}_{(0,\eta)}\big(y(t)^2\big)\geqslant\frac{1}{4}E\left(m(t)^2\right)\geqslant ct\tag{4.16}$$

holds, with $\mathbb{E}_{(0,\eta)}$ the distribution of the auxiliary process with initial condition $(0,\eta)$, E the distribution of the process $m(t)$ with initial condition $m(0)=0$ and c a strictly positive constant. This ends the proof of strict positivity of the right hand side in (4.10) and therefore of Theorem 4.2. ▮

5. SCALING LIMIT OF THE LOGARITHMIC SOBOLEV INEQUALITY

In this section, by a scaling limit of the logarithmic Sobolev inequality, we derive the exponential decrease of a suitable Lyapunov functional for a nonlinear degenerate parabolic equation, called porous media equation. In our context, since the density is bounded by one, this result can be obtained directly from the Poincaré inequality for the Laplacian. This procedure of taking scaling limit of a inequality for a microscopic dynamics might however give some non-trivial information when the density is unbounded, for instance in the so-called zero-range process for which a logarithmic Sobolev inequality has been recently proven.[9]

Let us consider the following parabolic problem, called porous media equation, on $B:=[0,1]^d$ with Dirichlet boundary conditions:

$$\begin{array}{ll} \partial_t u(t,r)=\nabla_r\cdot\big(D(u(t,r))\nabla_r u(t,r)\big) & (t,r)\in(0,\infty)\times B,\\ u(t,r)=\rho & (t,r)\in(0,\infty)\times\partial B,\\ u(0,r)=\varphi(r), & r\in B, \end{array} \tag{5.1}$$

where $\rho\in(0,1)$, the initial datum $\varphi\in C(B;[0,1])$ satisfies $\varphi(r)=\rho$ for $r\in\partial B$, and the diffusion coefficient $D(u)\geqslant 0$ is smooth and degenerates linearly for $u=1$, namely the exists a constant $\delta\in(0,1)$ such that $\delta(1-u)\leqslant D(u)\leqslant\delta^{-1}(1-u)$, $u\in[0,1]$. Since we assumed $0\leq\varphi\leqslant 1$, by the maximum principle, we have that $u\in C(\mathbb{R}_+\times B;[0,1])$. As discussed in the introduction, Eq. (5.1) is the natural candidate for the hydrodynamic limit of the process with generator $L_\Lambda^{(k)}$. Note that the Dirichlet boundary condition is due to the particles' reservoirs. In general, the diffusion coefficient $D(u)$ would be given by the Green–Kubo formula ref. 28, section II.2.2; however, if the rates in (2.7) are modified to

$$c_{x,x+e_i}(\eta)=\frac{1}{2}\big[2-\eta_{x-e_i}-\eta_{x+2e-i}\big] \tag{5.2}$$

the model becomes *gradient* ref. 28, II.2.4 and, in such a case, the (putative) diffusion coefficient would be simply given by $D(u)=(1-u)$.

Given $\rho \in (0,1)$, we introduce the convex functional $H_\rho : C(B;[0,1]) \to \mathbb{R}_+$ as

$$H_\rho(u) := \int_B dr \left[u(r) \log \frac{u(r)}{\rho} + (1-u(r)) \log \frac{1-u(r)}{1-\rho} \right], \tag{5.3}$$

where we understand $0 \log 0 = 0$. It is easy to show that H_ρ is a Lyapunov functional for the evolution (5.1), moreover if $u(t,r)$ is a smooth solution of (5.1) bounded away from 0 and 1 we have

$$\begin{aligned} -\frac{d}{dt} H_\rho(u(t,\cdot)) &= \int_B dr\, u(t,r)[1-u(t,r)] D(u(t,r)) \Big(\nabla_r \log \frac{u(t,r)}{1-u(t,r)} \Big)^2 \\ &=: \mathcal{Q}(u(t,\cdot)). \end{aligned} \tag{5.4}$$

Theorem 5.1 which states a "logarithmic Sobolev inequality" for the nonlinear evolution (5.1) is easily obtained as a scaling limit of (3.4).

Theorem 5.1. For each $\rho \in (0,1)$ and $\delta > 0$ there exists a constant $C' = C'(d,\delta,\rho)$ such that for any $u \in C^1\big(B;[0,1]\big)$ with $u(r) = \rho$ for $r \in \partial B$

$$H_\rho(u) \leqslant C' \mathcal{Q}(u). \tag{5.5}$$

The exponential decrease of the functional H_ρ along the flow of the porous media equation (5.1) follows from (5.4), Theorem 5.1, and a straightforward truncation argument.

Corollary 5.2. Let $u \in C(\mathbb{R}_+ \times B;[0,1])$ be the solution of (5.1) and $C' = C'(d,\delta,\rho)$ be the constant in (5.5). For each $\rho \in (0,1)$ we have

$$H_\rho(u(t,\cdot)) \leqslant e^{-t/C'} H_\rho(\varphi) \tag{5.6}$$

for any $t \in \mathbb{R}_+$ and any $\varphi \in C(B;[0,1])$ such that $\varphi(r) = \rho$ for $r \in \partial B$.

Remark. As mentioned before, inequality (5.5) can be proven directly by reducing it to the Poincaré inequality for the Dirichlet Laplacian on B. Indeed, since $0 \leqslant u \leqslant 1$ we have that $H_\rho(u)$ is equivalent to the norm of $u - \rho$ in $L_2(B, dr)$, more precisely for each $\rho \in (0,1)$ there exists a constant $A = A(\rho) > 0$ such that $A^{-1}\|u-\rho\|_2^2 \leqslant H_\rho(u) \leqslant A\|u-\rho\|_2^2$. Therefore (5.5) follows by using definition (5.3), the above bounds, Poincaré inequality for the Dirichlet Laplacian on B, and identity $\nabla u = u(1-u)\nabla(\log[\frac{u}{1-u}])$.

Proof of Theorem 5.1. In the following we prove bound (5.5) for $D(u)=1-u$, the general case follows by our hypothesis on the diffusion coefficient D. By truncation, it is also enough to prove (5.5) when u is a smooth function bounded away from 0 and 1.

We set $\varepsilon := \ell^{-1}$ and apply inequality (3.4) for the rates given in (5.2) and choose $f^2 = g_\varepsilon$ where

$$g_\varepsilon(\eta) = \prod_{x\in\Lambda} \frac{u(\varepsilon x)^{\eta_x}[1-u(\varepsilon x)]^{1-\eta_x}}{\rho^{\eta_x}[1-\rho]^{1-\eta_x}}. \tag{5.7}$$

Note that $\mu^\varepsilon_{\Lambda,\rho,u}(\eta) := \mu_{\Lambda,\rho}(\eta) g_\varepsilon(\eta)$ is a product probability measure on Ω_Λ with density profile u, namely $\mu^\varepsilon_{\Lambda,\rho,u}(\eta_x) = u(\varepsilon x)$. By elementary computations which we omit, we have that the normalized relative entropy of $\mu^\varepsilon_{\Lambda,\rho,u}$ w.r.t. $\mu_{\Lambda,\rho}$ converges to $H_\rho(u)$ as $\varepsilon \to 0$, namely

$$\lim_{\varepsilon\to 0} \varepsilon^d \mu_{\Lambda,\rho}\Big(g_\varepsilon \log \frac{g_\varepsilon}{\mu_{\Lambda,\rho}(g_\varepsilon)}\Big) = H_\rho(u). \tag{5.8}$$

Moreover it is straightforward to check that

$$\lim_{\varepsilon\to 0} \varepsilon^{d-2} \mathcal{E}_{\Lambda,\rho}(\sqrt{g_\varepsilon}) = \mathcal{Q}(u). \tag{5.9}$$

Let $C(d,1,\rho)$ be the constant such that (3.4) holds for the rates $c^{(1)}$; then (3.4) holds for the rates given in (5.2) with $2C(d,1,\rho)$. By (5.8) and (5.9) the bound (5.5), with $C' = 2C(d,1,\rho)$, now follows from Theorem 3.2. ∎

We have assumed that the diffusion coefficient $D(u)$ degenerates linearly for $u=1$. One can also obtain the exponential decrease of the Lyapunov functional H_ρ if $D(u) \asymp (1-u)^n$, n a positive integer. This can be shown by introducing a microscopic model in which the exchange rate $c_{x,x+e_i}(\eta)$ is zero iff there exists $j=1,\dots,n$ such that $\eta_{x-je_i} = \eta_{x+(j+1)e_i} = 1$. By arguments analogous to those in Section 3 it is in fact possible to prove that the logarithmic Sobolev constant for such a model is of the order ℓ^2.

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